

# Frequency Domain Controller Design

## 9.2 Frequency Response Characteristics

The frequency transfer functions are defined for sinusoidal inputs having all possible frequencies  $\omega \in [0, +\infty)$ . They are obtained from (9.1) by simply setting  $s = j\omega$ , that is

$$\text{Open-loop: } G(j\omega)H(j\omega), \quad \text{Closed-loop: } \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)} = M(j\omega) \quad (9.1)$$

Typical diagrams for the magnitude and phase of the open-loop frequency transfer function are presented in Figure 9.1.

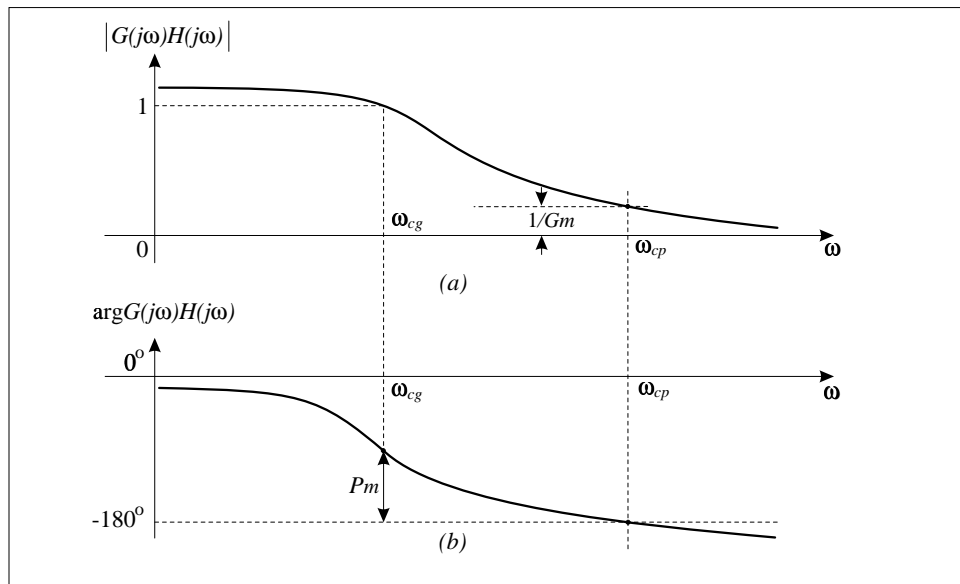


Figure 9.1: Magnitude (a) and phase (b) of the open-loop transfer function

**System Bandwidth:** This represents the frequency range in which the magnitude of the closed-loop frequency transfer function drops no more than 3 dB (decibels) from its zero-frequency value. The system bandwidth can be obtained from the next equality, which indicates the attenuation of 3 dB, as

$$|M(j\omega_{BW})| = \frac{1}{\sqrt{2}}|M(0)| \Rightarrow \omega_{BW} \quad (9.2)$$

**Peak Resonance:** This is obtained by finding the maximum of the function  $|M(j\omega)|$  with respect to frequency  $\omega$ . It is interesting to point out that the systems having large maximum overshoot have also large peak resonance. This is analytically justified for a second-order system in Problem 9.1.

**Resonant Frequency:** This is the frequency at which the peak resonance occurs. It can be obtained from

$$\frac{d}{d\omega}|M(j\omega)| = 0 \Rightarrow \omega_r$$

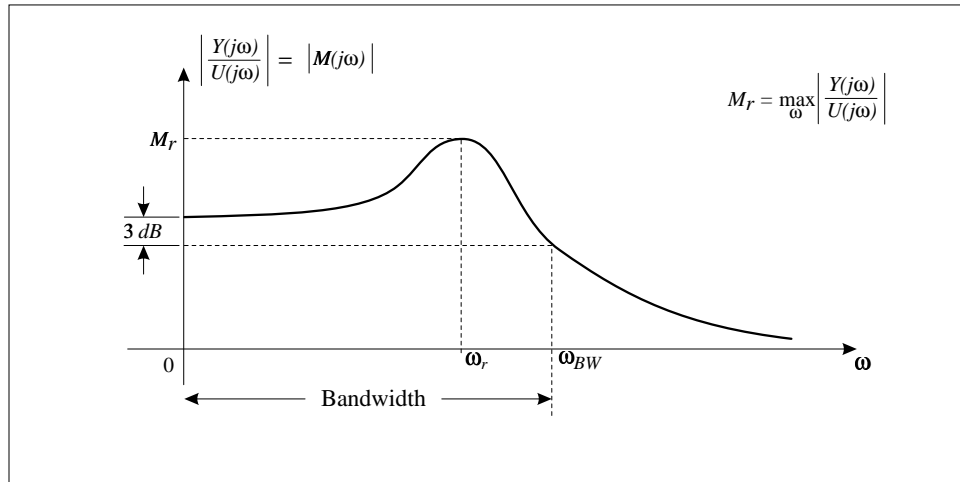


Figure 9.2: Magnitude of the closed-loop transfer function

### 9.3 Bode Diagrams

*Bode diagrams represent the frequency plots of the magnitude and phase of the open-loop frequency transfer function  $G(j\omega)H(j\omega)$ . The magnitude is plotted in dB (decibels) on the  $\log \omega$  scale. We first study independently the magnitude and frequency plots of each of these elementary frequency transfer functions. Since the open-loop frequency transfer function  $G(j\omega)H(j\omega)$  is given in terms of products and ratios of elementary transfer functions, it is easy to see that the phase of  $G(j\omega)H(j\omega)$  is obtained by summing and subtracting phases of the elementary transfer functions. Also, by expressing the magnitude of the open-loop transfer function in decibels, the magnitude  $|G(j\omega)H(j\omega)|_{dB}$  is obtained by adding the magnitudes of the elementary frequency transfer functions. For example*

$$\begin{aligned}
 |G(j\omega)H(j\omega)|_{dB} &= 20 \log_{10} \left| \frac{K(j\omega + z_1)(j\omega + z_2)}{(j\omega)(j\omega + p_2)(j\omega + p_3)} \right| \\
 &= 20 \log_{10} |K| + 20 \log_{10} |j\omega + z_1| + 20 \log_{10} |j\omega + z_2| \\
 &\quad + 20 \log_{10} \left| \frac{1}{j\omega} \right| + 20 \log_{10} \left| \frac{1}{j\omega + p_2} \right| + 20 \log_{10} \left| \frac{1}{j\omega + p_3} \right|
 \end{aligned}$$

and

$$\begin{aligned}
 \arg \{G(j\omega)H(j\omega)\} &= \arg \{K\} + \arg \{j\omega + z_1\} + \arg \{j\omega + z_2\} \\
 &\quad - \arg \{j\omega\} - \arg \{j\omega + p_2\} - \arg \{j\omega + p_3\}
 \end{aligned}$$

**Constant Term:** Since

$$K_{dB} = 20 \log_{10} K = \begin{cases} \text{positive number} & K > 1 \\ \text{negative number} & K < 1 \end{cases} \quad (9.3)$$

$$\arg K = \begin{cases} 0^\circ, & K > 0 \\ -180^\circ, & K < 0 \end{cases}$$

the magnitude and phase of this element are easily drawn and are presented in Figure 9.3.

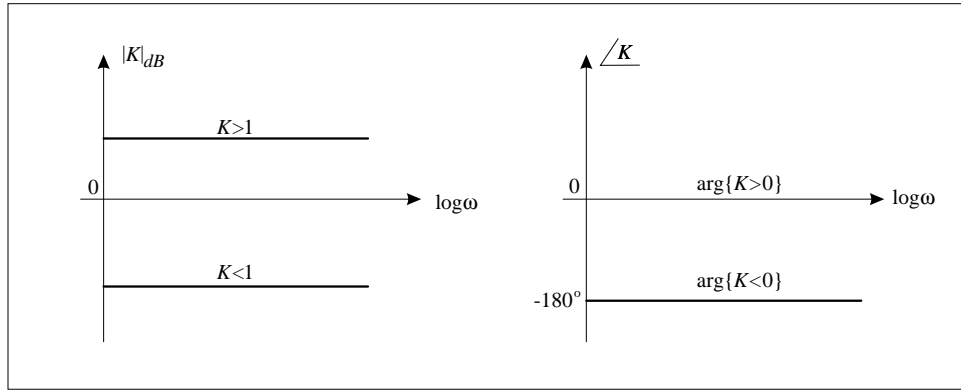


Figure 9.3: Magnitude and phase diagrams for a constant

**Pure Integrator:** The transfer function of a pure integrator, given by

$$G(j\omega) = \frac{1}{j\omega} \quad (9.4)$$

has the following magnitude and phase

$$|G(j\omega)|_{dB} = 20 \log_{10} \frac{1}{\omega} = -20 \log_{10} \omega, \quad \arg G(j\omega) = -90^\circ \quad (9.5)$$

It can be observed that the phase for a pure integrator is constant, whereas the magnitude is represented by a straight line intersecting the frequency axis at  $\omega = 1$  and having the slope of  $-20$  dB/decade. Both diagrams are represented in Figure 9.4. Thus, *a pure integrator introduces a phase shift of  $-90^\circ$  and a gain attenuation of  $-20$  dB/decade.*

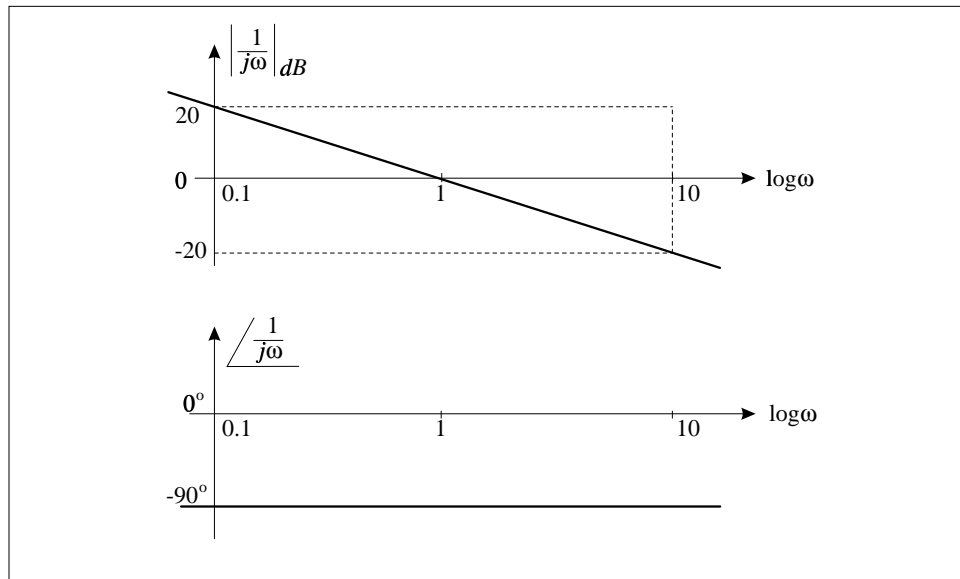


Figure 9.4: Magnitude and phase diagrams for a pure integrator

**Pure Differentiator:** The transfer function of a pure differentiator is given by

$$G(j\omega) = j\omega \quad (9.6)$$

Its magnitude and phase are easily obtained as

$$|G(j\omega)|_{dB} = 20 \log_{10} \omega, \quad \arg G(j\omega) = 90^\circ \quad (9.7)$$

The corresponding frequency diagrams are presented in Figure 9.5. *It can be concluded that a pure differentiator introduces a positive phase shift of  $90^\circ$  and an amplification of 20 dB/decade.*

**Real Pole:** The transfer function of a real pole, given by

$$G(j\omega) = \frac{p}{p + j\omega} = \frac{1}{1 + j\frac{\omega}{p}} \quad (9.8)$$

has the following magnitude and phase

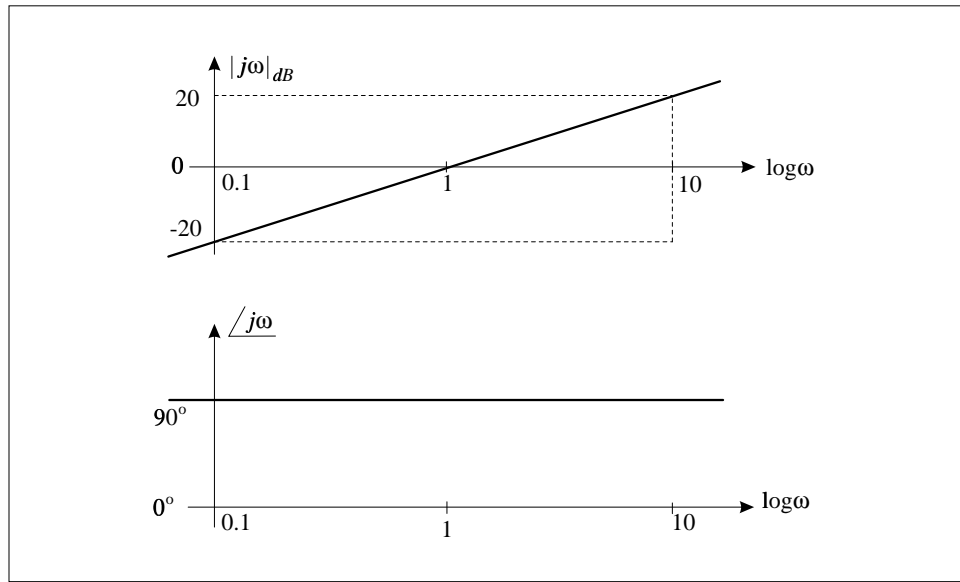


Figure 9.5: Magnitude and phase diagrams for a pure differentiator

$$|G(j\omega)|_{dB} = -20 \log_{10} \left[ 1 + \left( \frac{\omega}{p} \right)^2 \right]^{1/2}, \quad \arg G(j\omega) = -\tan^{-1} \left( \frac{\omega}{p} \right) \quad (9.9)$$

The phase diagram for a real pole can be plotted directly from (9.10). It can be seen that for large values of  $\omega$ ,  $\omega \gg p$ , the phase contribution is  $-90^\circ$ . For  $\omega$  small,  $\omega \ll p$ , the phase is close to zero, and for  $\omega = p$  the phase contribution is  $-45^\circ$ . This information is sufficient to sketch  $\arg G(j\omega)$  as given in Figure 9.6.

For the magnitude, we see from (9.10) that for small  $\omega$  the magnitude is very close to zero. For large values of  $\omega$  we can neglect 1 compared to  $\omega/p$  so that we have a similar result as for a pure integrator, i.e. we obtain an attenuation of 20 dB/decade. For small and large frequencies we have straight-line approximations. These straight lines intersect at  $\omega = p$ , which is also known as a *corner frequency*. The actual magnitude curve is below the straightline approximations. It has the biggest deviation from the asymptotes at the corner frequency (see Figure 9.6).

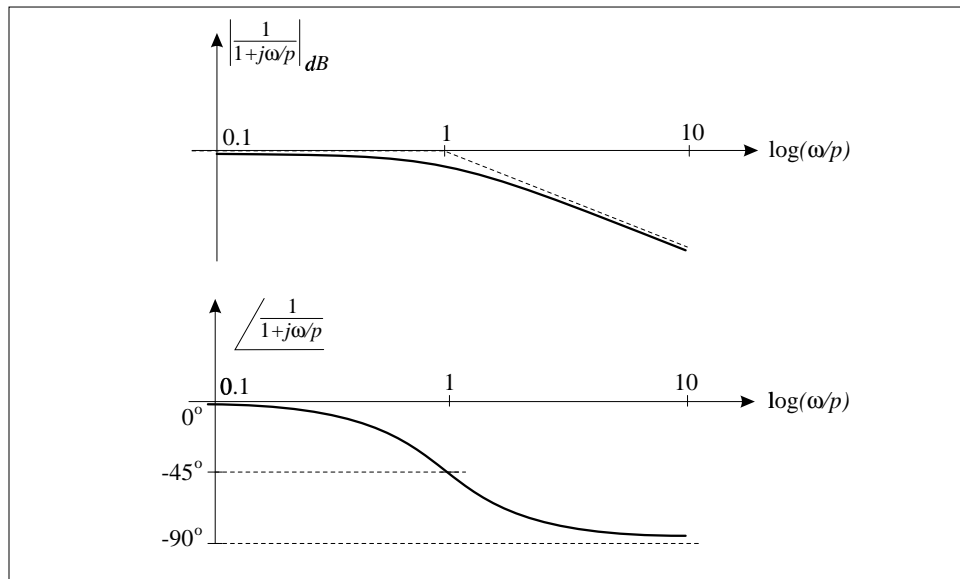


Figure 9.6: Magnitude and phase diagrams for a real pole

**Real Zero:** The transfer function of an element representing a real zero is

$$G(j\omega) = \frac{1}{z}(z + j\omega) = 1 + j\left(\frac{\omega}{z}\right) \quad (9.10)$$

Its magnitude and phase are

$$|G(j\omega)|_{dB} = 20 \log_{10} \left[ 1 + \left( \frac{\omega}{z} \right)^2 \right]^{1/2}, \quad \arg G(j\omega) = \tan^{-1} \left( \frac{\omega}{z} \right) \quad (9.11)$$

For small frequencies an asymptote for the magnitude is equal to zero and for large frequencies the magnitude asymptote has a slope of 20 dB/decade and intersects the real axis at  $\omega = z$  (the corner frequency). The phase diagram for small frequencies also has an asymptote equal to zero and for large frequencies an asymptote of  $90^\circ$ .

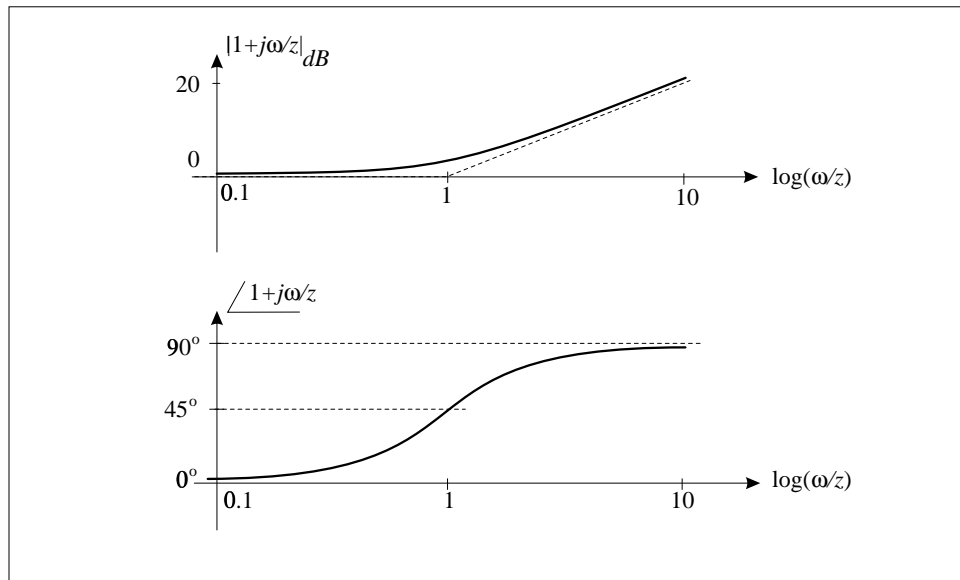


Figure 9.7: Magnitude and phase diagrams for a real zero



**Complex Conjugate Poles:** The transfer function is given by

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\frac{\omega}{\omega_n}} \quad (9.12)$$

The magnitude and phase of this second-order system are given by

$$|G(j\omega)|_{dB} = -20 \log_{10} \left[ \left( \frac{2\zeta\omega}{\omega_n} \right)^2 + \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 \right]^{1/2} \quad (9.13)$$

$$\arg G(j\omega) = -\tan^{-1} \left( \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \right)$$

For large values of  $\omega$  the corresponding approximations of (9.14) are

$$|G(j\omega)|_{dB} \approx -20 \log_{10} \left( \frac{\omega^2}{\omega_n^2} \right) = -40 \log_{10} \left( \frac{\omega}{\omega_n} \right)$$

$$\arg \{G(j\omega)\} \approx -\tan^{-1} \left( \frac{2\zeta\omega_n}{-\omega} \right) \rightarrow -\tan^{-1} (0^-) = -180^\circ$$

At low frequencies the approximations can be obtained directly from (9.13), that is

$$G(j\omega) \approx \frac{\omega_n^2}{\omega_n^2} = 1 \Rightarrow |G(j\omega)|_{dB} = 0, \quad \arg \{G(j\omega)\} = 0^\circ$$

The asymptotes for small and large frequencies are, respectively, zero and  $-40\text{dB/decade}$  (with the corner frequency at  $\omega = \omega_n$ ) for the magnitude, and zero and  $-180^\circ$  for the phase. At the corner frequency  $\omega_n$  the phase is equal to

$-90^\circ$ . Note that the actual plot in the neighborhood of the corner frequency depends on the values of the damping ratio  $\zeta$ . Several curves are shown for  $0.1 \leq \zeta \leq 1$ . It can be seen from Figure 9.8 that the smaller  $\zeta$ , the higher peak.

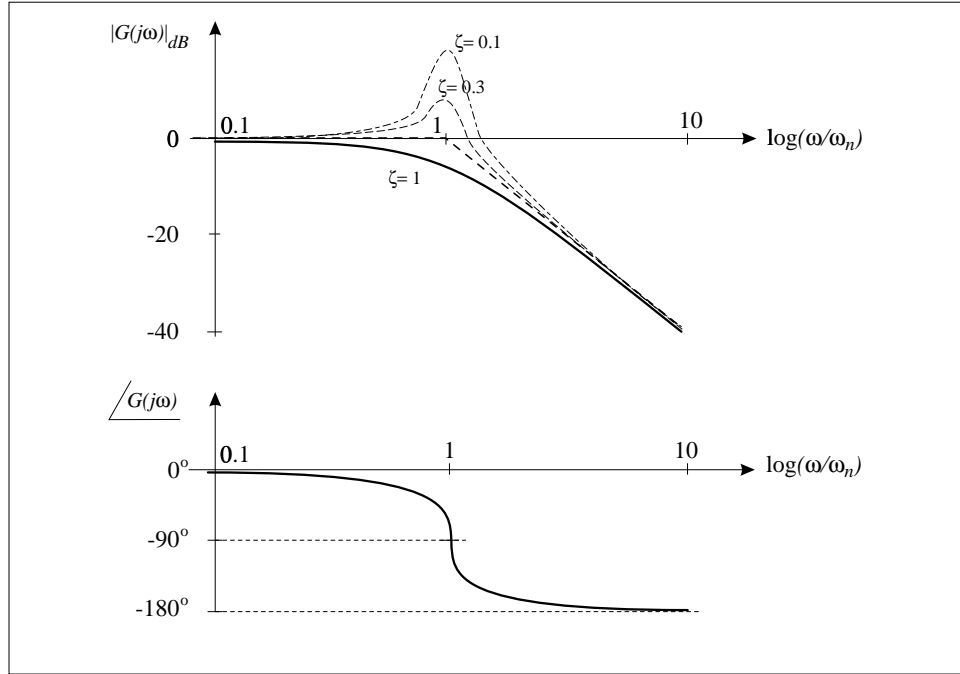


Figure 9.8: Magnitude and phase diagrams for complex conjugate poles

**Complex Conjugate Zeros:** The complex conjugate zero is given by

$$G(j\omega) = 1 + 2\zeta j \left( \frac{\omega}{\omega_n} \right) + \left( \frac{j\omega}{\omega_n} \right)^2 = 1 - \left( \frac{\omega}{\omega_n} \right)^2 + j2\zeta \left( \frac{\omega}{\omega_n} \right) \quad (9.14)$$

so that the corresponding Bode diagrams will be the mirror images of the Bode diagrams obtained for the complex conjugate poles represented by (9.13). Here, the asymptotes for small frequencies are equal to zero for both the magnitude and phase plots; for high frequencies the magnitude asymptote has a slope of 40 dB/decade and starts at the corner frequency of  $\omega = \omega_n$ , and the phase plot asymptote is  $180^\circ$ .

### 9.3.1 Phase and Gain Stability Margins from Bode Diagrams

Bearing in mind the definition of the phase and gain stability margins given in (4.54) and (4.55), and the corresponding phase and gain crossover frequencies defined in (4.56) and (4.57), it is easy to conclude that these margins can be found from Bode diagrams as indicated in Figure 9.9.

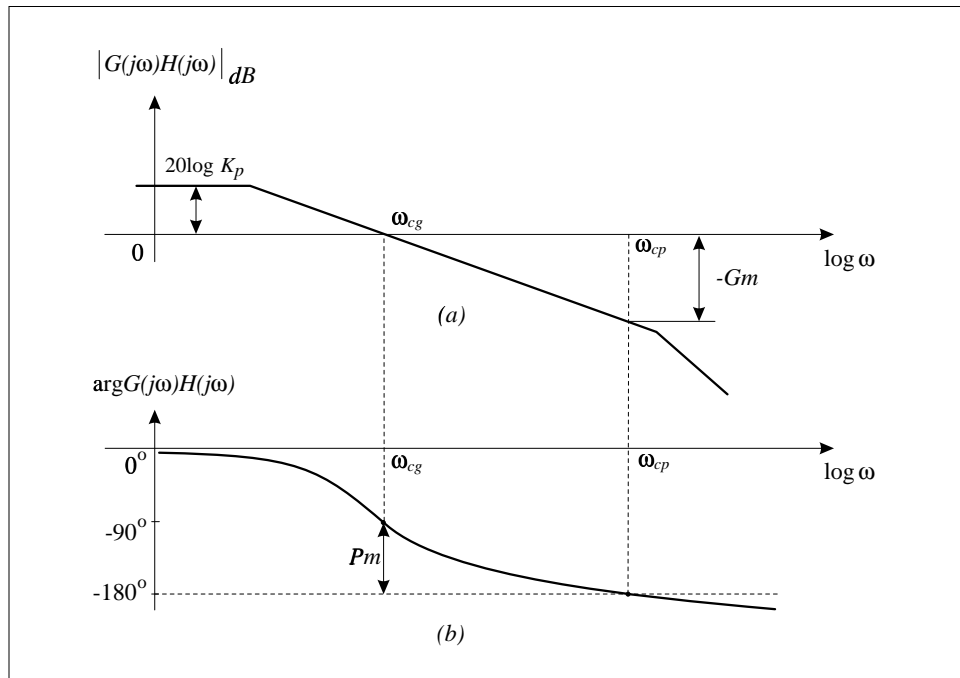


Figure 9.9: Gain and phase margins and Bode diagrams

**Example 9.1:** Consider the open-loop frequency transfer function

$$G(j\omega)H(j\omega) = \frac{(j\omega + 1)}{j\omega(j\omega + 2) \left[ (j\omega)^2 + 2(j\omega) + 2 \right]}$$

Using MATLAB

```
num=[1 1];d1=[1 0];d2=[1 2];d3=[1 2 2];
den1=conv(d1,d2); den=conv(den1,d3);
bode(num,den);[Gm,Pm,wcp,wcg]=margin(num,den);
```

The phase and gain stability margins and the crossover frequencies are

$$Gm = 8.9443 \text{ dB}, Pm = 82.2462^\circ, \omega_{cp} = 1.7989 \text{ rad/s}, \omega_{cg} = 0.2558 \text{ rad/s}$$

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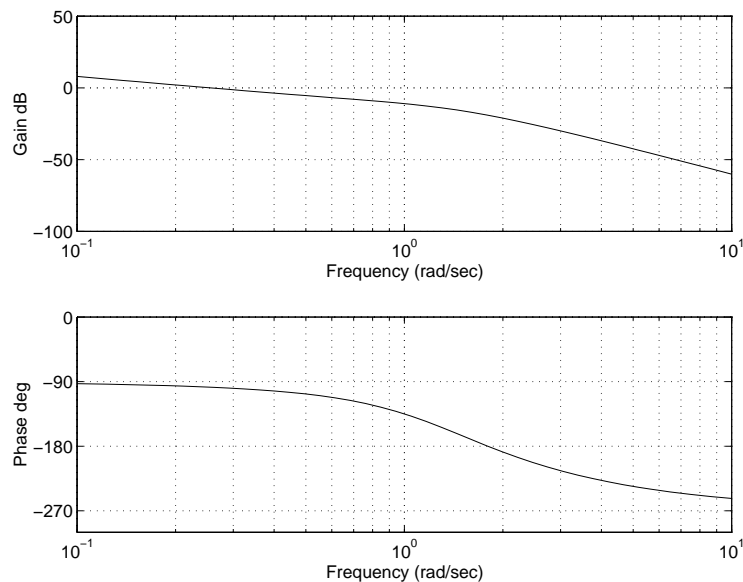


Figure 9.10: Bode diagrams for Example 9.1

### 9.3.2 Steady State Errors and Bode Diagrams

Steady state errors can be indirectly determined from Bode diagrams by reading the values for constants  $K_p, K_v, K_a$  from them. Knowing these constants, the corresponding errors are easily found by using formulas (6.30), (6.32), and (6.34). The steady state errors and corresponding constants  $K_p, K_v, K_a$  are first of all determined by the system type, which represents the multiplicity of the pole at the origin of the open-loop feedback transfer function, in general, represented by

$$G(j\omega)H(j\omega) = \frac{K(j\omega + z_1)(j\omega + z_2) \cdots}{(j\omega)^r(j\omega + p_1)(j\omega + p_2) \cdots} \quad (9.15)$$

This can be rewritten as

$$\begin{aligned} G(j\omega)H(j\omega) &= \frac{K z_1 z_2 \cdots \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right) \cdots}{p_1 p_2 \cdots (j\omega)^r \left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right) \cdots} \\ &= \frac{K_B \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right) \cdots}{(j\omega)^r \left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right) \cdots} \end{aligned} \quad (9.16)$$

where

$$K_B = \frac{K z_1 z_2 \cdots}{p_1 p_2 \cdots} \quad (9.17)$$

is known as *Bode's gain*, and  $r$  is the type of feedback control system.

For control systems of type  $r = 0$ , the position constant according to formula (6.31) is obtained from (9.17) as

$$K_p = \frac{K_B \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right) \dots}{(j\omega)^0 \left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right) \dots} \Big|_{j\omega=0} = K_B \quad (9.18)$$

It follows from (9.17)–(9.19) that the corresponding magnitude Bode diagram of type zero control systems for small values of  $\omega$  is flat (has a slope of 0 dB) and the value of  $20 \log K_B = 20 \log K_p$ . This is graphically represented in Figure 9.12.

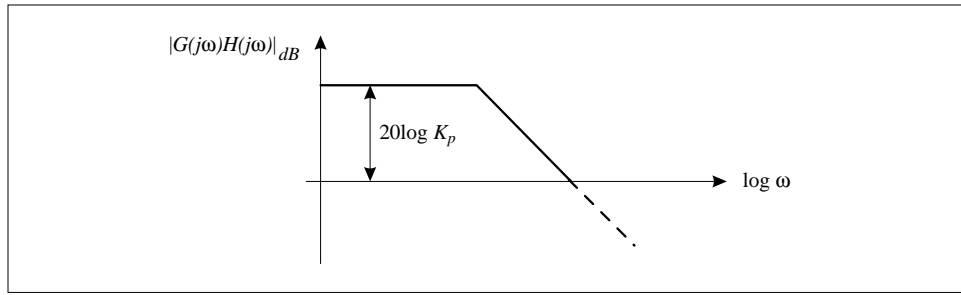


Figure 9.12: Magnitude Bode diagram of type zero control systems at small frequencies

For control systems of type  $r = 1$ , the open-loop frequency transfer function is approximated at low frequencies by

$$\frac{K_B \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right) \dots}{(j\omega)^1 \left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right) \dots} \approx \frac{K_B}{(j\omega)^1} \quad (9.19)$$

It follows that the corresponding magnitude Bode diagram of type one control systems for small values of  $\omega$  has a slope of  $-20$  dB/decade and the values of

$$20 \log \left| \frac{K_B}{j\omega} \right| = 20 \log |K_B| - 20 \log |\omega| \quad (9.20)$$

From (9.20) and (6.33) it is easy to conclude that for type one control systems the velocity constant is  $K_v = K_B$ . Using this fact and the frequency plot of (9.21), we conclude that  $K_v$  is equal to the frequency  $\omega^*$  at which the line (9.21) intersects the frequency axis, that is

$$0 = 20 \log |K_B| - 20 \log |\omega^*| \Rightarrow K_B = \omega^* = K_v \quad (9.21)$$

This is graphically represented in Figure 9.13.

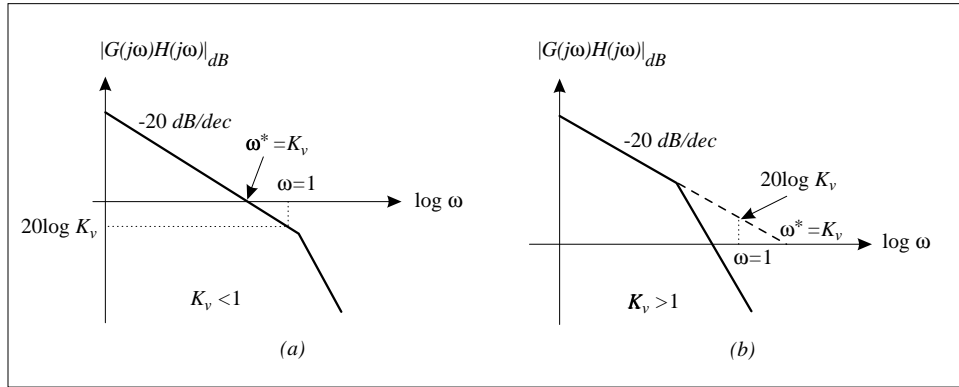


Figure 9.13: Magnitude Bode diagram of type one control systems at small frequencies

Note that if  $K_v = \omega^* > 1$ , the corresponding frequency  $\omega^*$  is obtained at the point where the extended initial curve, which has a slope of  $-20$  dB/decade, intersects the frequency axis (see Figure 9.13b).

Similarly, for type two control systems,  $r = 2$ , we have at low frequencies

$$\frac{K_B \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right) \cdots}{(j\omega)^2 \left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right) \cdots} \approx \frac{K_B}{(j\omega)^2} \quad (9.22)$$

which indicates an initial slope of  $-40$  dB/decade and a frequency approximation of

$$20 \log \left| \frac{K_B}{(j\omega)^2} \right| = 20 \log |K_B| - 20 \log |\omega^2| = 20 \log |K_B| - 40 \log \omega \quad (9.23)$$

From (9.23) and (6.35) it is easy to conclude that for type two control systems the acceleration constant is  $K_a = K_B$ . From the frequency plot of the straight line (9.24), it follows that  $K_B = (\omega^{**})^2$ , where  $\omega^{**}$  represents the intersection of the initial magnitude Bode plot with the frequency axis as represented in Figure 9.14.

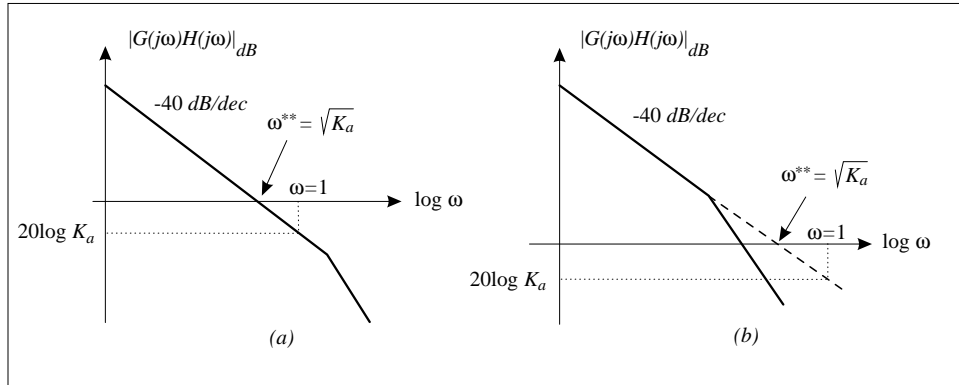


Figure 9.14: Magnitude Bode diagram of type two control systems at small frequencies

It can be seen from Figures 9.12–9.14 that by increasing the values for the magnitude Bode diagrams at low frequencies (i.e. by increasing  $K_B$ ), the constants  $K_p$ ,  $K_v$ , and  $K_a$  are increased. According to the formulas for steady state errors, given in (6.30), (6.32), and (6.34) as

$$e_{ss_{step}} = \frac{1}{1 + K_p}, \quad e_{ss_{ramp}} = \frac{1}{K_v}, \quad e_{ss_{parabolic}} = \frac{2}{K_a}$$



we conclude that in this case the steady state errors are decreased. Thus, *the bigger  $K_B$ , the smaller the steady state errors*.

**Example 9.3:** Consider Bode diagrams obtained in Examples 9.1 and 9.2. The Bode diagram in Figure 9.10 has an initial slope of  $-20$  dB/decade which intersects the frequency axis at roughly  $\omega^* = 0.2$  rad/s. Thus, we have for the Bode diagram in Figure 9.10

$$K_p = \infty, \quad K_v \approx 0.2, \quad K_a = 0$$

Using the exact formula for  $K_v$ , given by (6.33), we get

$$K_v = \lim_{s \rightarrow 0} \left\{ s \frac{(s+1)}{s(s+2)(s^2+2s+2)} \right\} = 0.25$$

In Figure 9.11 the initial slope is 0 dB, and hence we have from this diagram

$$20 \log K_p \approx 15 \Rightarrow K_p \approx 5.62, \quad K_v = 0, \quad K_a = 0$$

Using the exact formula for  $K_p$  as given by (6.31) produces

$$K_p = \lim_{s \rightarrow 0} \left\{ \frac{10(s+4)}{(s+1)(s+2)(s+3)} \right\} = 6.67$$

Note that the accurate results about steady state error constants are obtained easily by using the corresponding formulas; hence the Bode diagrams are used only for quick and rough estimates of these constants.

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## 9.4 Compensator Design Using Bode Diagrams

*Controller design techniques in the frequency domain will be governed by the following facts:*

- (a) *Steady state errors are improved by increasing Bode's gain  $K_B$ .*
- (b) *System stability is improved by increasing phase and gain margins.*
- (c) *Overshoot is reduced by increasing the phase stability margin.*
- (d) *Rise time is reduced by increasing the system's bandwidth.*

The first two items, (a) and (b), have been already clarified. In order to justify item (c), we consider the open-loop transfer function of a second-order system

$$G(j\omega)H(j\omega) = \frac{\omega_n^2}{(j\omega)(j\omega + 2\zeta\omega_n)} \quad (9.24)$$

whose gain crossover frequency can be easily found from

$$|G(j\omega_{cg})H(j\omega_{cg})| = \frac{\omega_n^2}{\omega \sqrt{\omega^2 + 4\zeta^2\omega_n^2}} = 1 \quad (9.25)$$

leading to

$$\omega_{cg} = \omega_n \sqrt{\sqrt{1 + 4\zeta^2} - 2\zeta^2} \quad (9.26)$$

The phase of (9.25) at the gain crossover frequency is

$$\arg \{G(j\omega_{cg})H(j\omega_{cg})\} = -90^\circ - \tan^{-1} \frac{\omega_{cg}}{2\zeta\omega_n} \quad (9.27)$$

so that the corresponding phase margin becomes

$$Pm = \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{1 + 4\zeta^2} - 2\zeta^2}} = Pm(\zeta) \quad (9.28)$$

Plotting the function  $Pm(\zeta)$ , it can be shown that it is a monotonically increasing function with respect to  $\zeta$ ; we therefore conclude that *the higher phase margin, the larger the damping ratio, which implies the smaller the overshoot*.

Item (d) cannot be analytically justified since we do not have an analytical expression for the response rise time. However, it is very well known from undergraduate courses on linear systems and signals that rapidly changing signals have a wide bandwidth. Thus, *systems that are able to accommodate fast signals must have a wide bandwidth*.

### 9.4.1 Phase-Lag Controller

The transfer function of a phase-lag controller is given by

$$G_{lag}(j\omega) = \left(\frac{p_1}{z_1}\right) \frac{z_1 + j\omega}{p_1 + j\omega} = \frac{1 + j\frac{\omega}{z_1}}{1 + j\frac{\omega}{p_1}}, \quad z_1 > p_1 \quad (9.29)$$

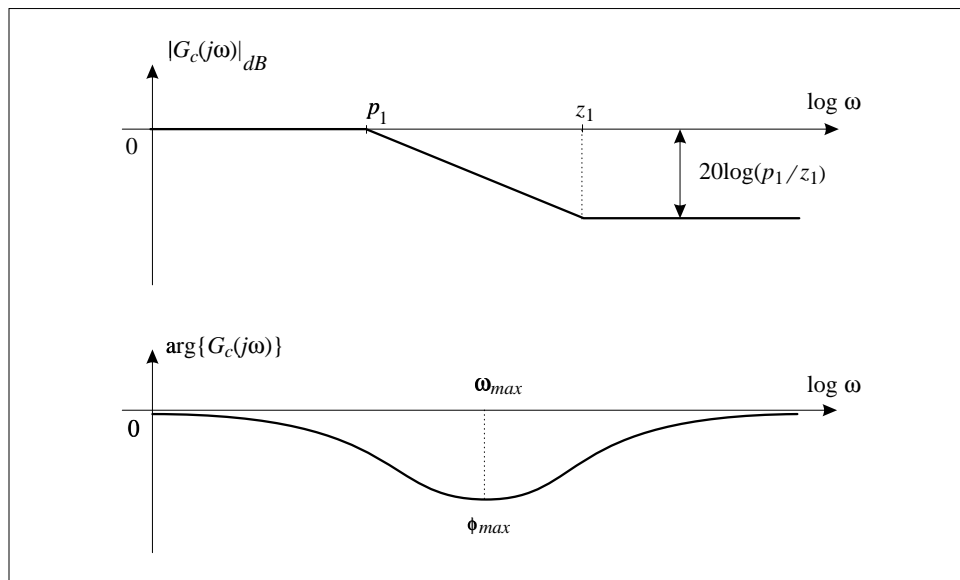


Figure 9.15: Magnitude approximation and exact phase of a phase-lag controller

Due to attenuation of the phase-lag controller at high frequencies, the frequency bandwidth of the compensated system (controller and system in series) is reduced. Thus, *the phase-lag controllers are used in order to decrease the system bandwidth (to slow down the system response)*. In addition, they can be used *to improve the stability margins (phase and gain) while keeping the steady state errors constant*.

Expressions for  $\omega_{max}$  and  $\phi_{max}$  of a phase-lag controller will be derived in the next subsection in the context of the study of a phase-lead controller.

### 9.4.2 Phase-Lead Controller

The transfer function of a phase-lead controller is

$$G_{lead}(j\omega) = \left(\frac{p_2}{z_2}\right) \frac{z_2 + j\omega}{p_2 + j\omega} = \frac{1 + j\frac{\omega}{z_2}}{1 + j\frac{\omega}{p_2}}, \quad p_2 > z_2 \quad (9.30)$$

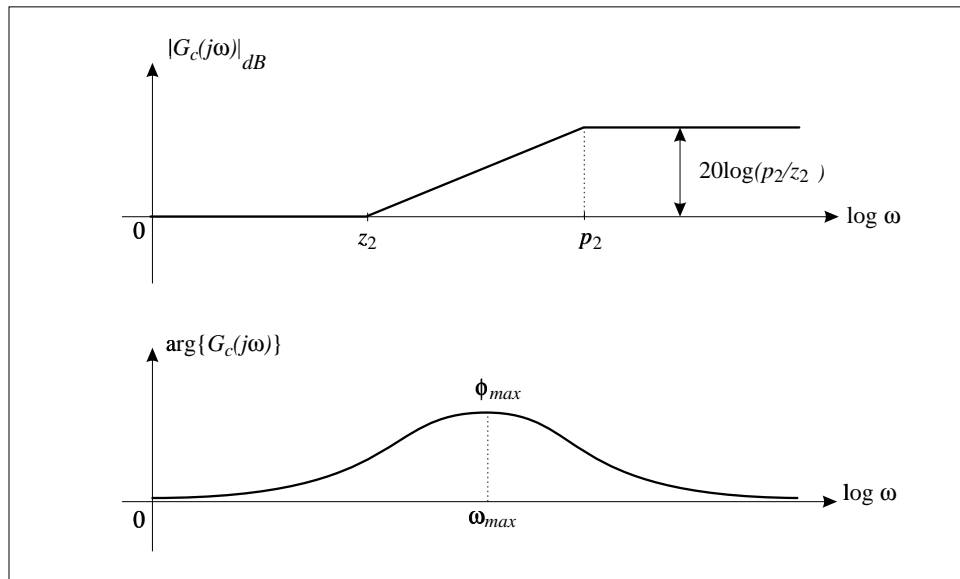


Figure 9.16: Magnitude approximation and exact phase of a phase-lead controller

Due to phase-lead controller (compensator) amplification at higher frequencies, it increases the bandwidth of the compensated system. *The phase-lead controllers are used to improve the gain and phase stability margins and to increase the system bandwidth (decrease the system response rise time).*

It follows from (9.31) that the phase of a phase-lead controller is given by

$$\arg \{G_{lead}(j\omega)\} = \tan^{-1} \left( \frac{\omega}{z_2} \right) - \tan^{-1} \left( \frac{\omega}{p_2} \right) \quad (9.31)$$

so that

$$\frac{d}{d\omega} \arg \{G_{lead}(j\omega)\} = 0 \Rightarrow \omega_{max} = \sqrt{z_2 p_2} \quad (9.32)$$

Assume that

$$p_2 = az_2, \quad a > 1 \quad \Rightarrow \quad \omega_{max} = \frac{p_2}{\sqrt{a}} \quad (9.33)$$

Substituting  $\omega_{max}$  in (9.32) implies

$$\tan \phi_{max} = \frac{a - 1}{2\sqrt{a}} \quad (9.34)$$

The parameter  $a$  can be found from (9.35) in terms of  $\phi_{max}$  as

$$a = \frac{1 + \sin \phi_{max}}{1 - \sin \phi_{max}} \quad (9.35)$$

Note that the same formulas for  $\omega_{max}$ , (9.33), and the parameter  $a$ , (9.36), hold for a *phase-lag controller* with  $p_1, z_1$  replacing  $p_2, z_2$  and with  $p_1 = az_1$ ,  $a < 1$ .

### 9.4.3 Phase-Lag-Lead Controller

The phase-lag-lead controller has the features of both phase-lag and phase-lead controllers and can be used to improve both the transient response and steady state errors. The frequency transfer function of the phase-lag-lead controller is given by

$$G_c(j\omega) = \frac{(j\omega + z_1)(j\omega + z_2)}{(j\omega + p_1)(j\omega + p_2)} = \frac{z_1 z_2}{p_1 p_2} \frac{\left(1 + j\frac{\omega}{z_1}\right)\left(1 + j\frac{\omega}{z_2}\right)}{\left(1 + j\frac{\omega}{p_1}\right)\left(1 + j\frac{\omega}{p_2}\right)} \quad (9.36)$$

$$= \frac{\left(1 + j\frac{\omega}{z_1}\right)\left(1 + j\frac{\omega}{z_2}\right)}{\left(1 + j\frac{\omega}{p_1}\right)\left(1 + j\frac{\omega}{p_2}\right)}, \quad z_1 z_2 = p_1 p_2, \quad p_2 > z_2 > z_1 > p_1$$

The Bode diagrams of this controller are shown in Figure 9.17.

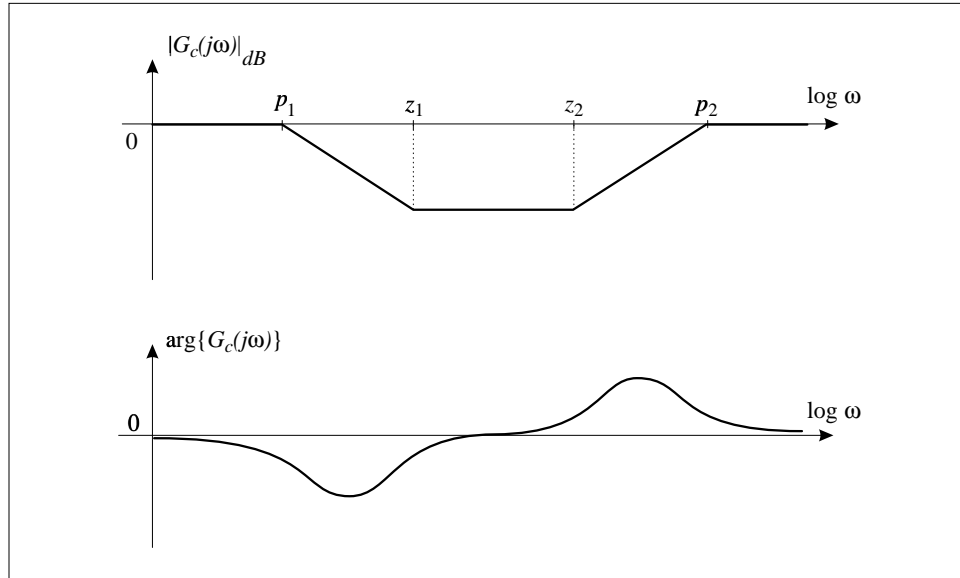


Figure 9.17: Bode diagrams of a phase-lag-lead controller

#### 9.4.4 Compensator Design with Phase-Lead Controller

The following algorithm can be used to design a controller (compensator) with a phase-lead network.

**Algorithm 9.1:**

1. Determine the value of the Bode gain  $K_B$  given by (9.18) as

$$K_B = \frac{K z_1 z_2 \cdots}{p_1 p_2 \cdots}$$

such that the steady state error requirement is satisfied.

2. Find the phase and gain margins of the original system with  $K_B$  determined in step 1.
3. Find the phase difference,  $\Delta\phi$ , between the actual and desired phase margins and take  $\phi_{max}$  to be  $5^\circ$ – $10^\circ$  greater than this difference. Only in rare cases should this be greater than  $10^\circ$ . This is due to the fact that we have to give an estimate of a new gain crossover frequency, which can not be determined very accurately (see step 5).
4. Calculate the value for parameter  $a$  from formula (9.36), i.e. by using

$$a = \frac{1 + \sin \phi_{max}}{1 - \sin \phi_{max}} > 1$$

5. Estimate a value for a compensator's pole such that  $\omega_{max}$  is roughly located at the new gain crossover frequency,  $\omega_{max} \approx \omega_{cgnew}$ . As a rule of thumb, add the gain of  $\Delta G = 20\log(a)$ [dB] at high frequencies to the uncompensated system and estimate the intersection of the magnitude diagram with the frequency axis, say  $\omega_1$ . The new gain crossover frequency is somewhere in between the old  $\omega_{cg}$  and  $\omega_1$ . Some authors (Kuo, 1991) suggest fixing the new gain crossover

frequency at the point where the magnitude Bode diagram has the value of  $-0.5\Delta G[\text{dB}]$ . Using the value for parameter  $a$  obtained in step 4 find the value for the compensator pole from (9.34) as  $-p_c = -\omega_{max}\sqrt{a}$  and the value for compensator's zero as  $-z_c = -p_c/a$ . Note that one can also guess a value for  $p_c$  and then evaluate  $z_c$  and  $\omega_{max}$ . The phase-lead compensator now can be represented by

$$G_c(s) = \frac{as + p_c}{s + p_c}$$

6. Draw the Bode diagram of the given system with controller and check the values for the gain and phase margins. If they are satisfactory, the controller design is done, otherwise repeat steps 1–5.

**Example 9.4:** Consider the following open-loop frequency transfer function

$$G(j\omega)H(j\omega) = \frac{K(j\omega + 6)}{(j\omega + 1)(j\omega + 2)(j\omega + 3)}$$

*Step 1.* Let the design requirements be set such that the steady state error due to a unit step is less than 2% and the phase margin is at least  $45^\circ$ . Since

$$e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + K_B}, \quad K_B = \frac{K \times 6}{1 \times 2 \times 3} = K$$

we conclude that  $K \geq 50$  will satisfy the steady state error requirement of being less than 2%. We know from the root locus technique that high static gains can damage system stability, and so for the rest of this design problem we take  $K = 50$ .

*Step 2.* We draw Bode diagrams of the uncompensated system with the Bode gain obtained in step 1 and determine the phase and gain margins and the crossover frequencies. This can be done via MATLAB



```

[den]=[1 6 11 6];
[num]=[50 300];
[Gm,Pm,wcp,wcg]=margin(num,den);
bode(num,den)

```

The corresponding Bode diagrams are presented in Figure 9.18a. The phase and gain margins are obtained as  $Gm = \infty$ ,  $Pm = 5.59^\circ$  and the crossover frequencies are  $\omega_{cg} = 7.5423 \text{ rad/s}$ ,  $\omega_{cp} = \infty$ .

*Step 3.* Since the desired phase is well above the actual one, the phase-lead controller must make up for  $45^\circ - 5.59^\circ = 39.41^\circ$ . We add  $10^\circ$ , for the reason explained in step 3 of Algorithm 9.1, so that  $\phi_{max} = 49.41^\circ$ .

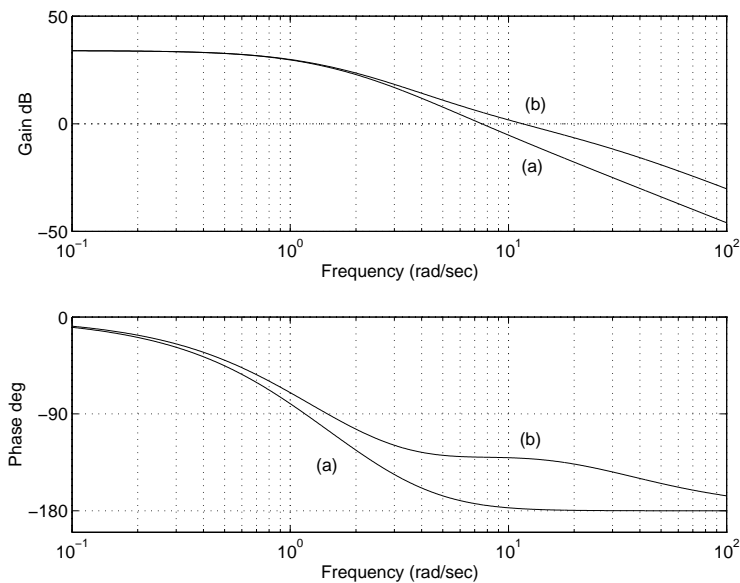


Figure 9.18: Bode diagrams for the original system (a) and compensated system (b) of Example 9.4

The above operations can be achieved by using MATLAB as follows

```
% estimate Phimax with Pmd = desired phase margin;
Pmd=input('enter desired value for phase margin');
Phimax=Pmd-Pm+10;
% converts Phimax into radians;
Phirad=(Phimax/180)*pi;
```

*Step 4.* Here we evaluate the parameter  $a$  according to the formula (9.36) and get  $a = 7.3144$ . This can be done in MATLAB by

```
a=(1+sin(Phirad))/(1-sin(Phirad));
```

*Step 5.* In order to obtain an estimate for the new gain crossover frequency we first find the controller amplification at high frequencies, which is equal to  $20\log(a) = 17.2836 \text{ dB} = \Delta G_{dB}$ . The magnitude Bode diagram increased by  $\Delta G_{dB}$  at high frequencies intersects the frequency axis at  $\omega_{max} \approx 10.5 \text{ rad/s}$ . We guess (estimate) the value for  $p_c$  as  $p_c = 25$ , which is roughly equal to  $\omega_{max}\sqrt{a}$ . By using  $p_c = 25$  and forming the corresponding compensator, we get for the compensated system  $Pmc = 48.2891^\circ$  at  $\omega_{cgnw} = 13.8519 \text{ rad/s}$ , which is satisfactory. This step can be performed by MATLAB as follows.

```
% Find amplification at high frequencies, DG;
DG=20*log10(a);
% estimate value for pole -pc from Step 5;
pc=input('enter estimated value for pole pc');
% form compensator's numerator;
nc=[a pc];
% form compensator's denominator;
dc=[1 pc];
% find the compensated system transfer function;
```

```

numc=conv(num,nc);
denc=conv(den,dc);
[Gmc,Pmc,wcp,wcg]=margin(numc,denc);
bode(numc,denc)

```

The phase-lead compensator obtained is given by

$$G_c(s) = \frac{7.3144s + 25}{s + 25} = \frac{as + p_2}{s + p_2}$$

*Step 6.* The Bode diagrams of the compensated control system are presented in Figure 9.18b. Both requirements are satisfied, and therefore the controller design procedure is successfully completed.

Note that num, den, numc, denc represent, respectively, the numerators and denominators of the open-loop transfer functions of the original and compensated systems. In order to find the corresponding closed-loop transfer functions, we use the MATLAB function `cloop`, that is

```

[cnum,cden]=cloop(num,den,-1);
% -1 indicates a negative unit feedback
[cnumc,cdenc]=cloop(numc,denc,-1);

```

The closed-loop step responses are obtained by

```

[y,x]=step(cnum,cden);
[yc,xc]=step(cnumc,cdenc);

```

and are represented in Figure 9.19. It can be seen from this figure that both the maximum percent overshoot and the settling time are drastically reduced. In addition, the rise time of the compensated system is shortened since the phase-lead controller increases the frequency bandwidth of the system.

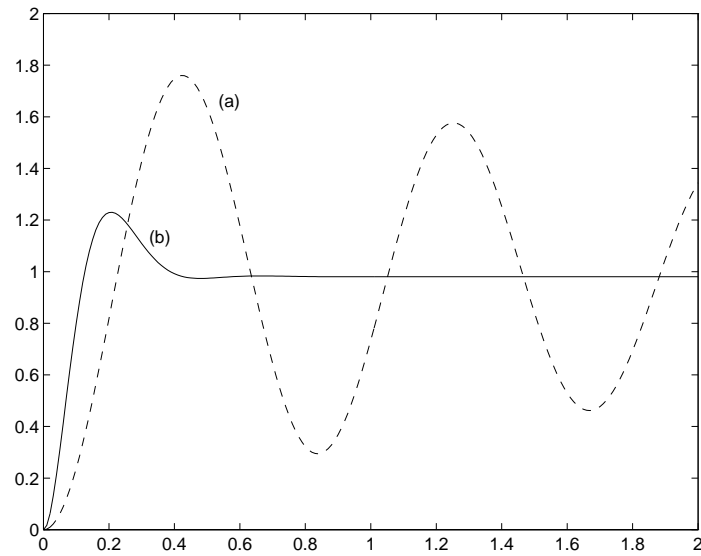


Figure 9.19: Step responses for the original (a) and compensated (b) systems

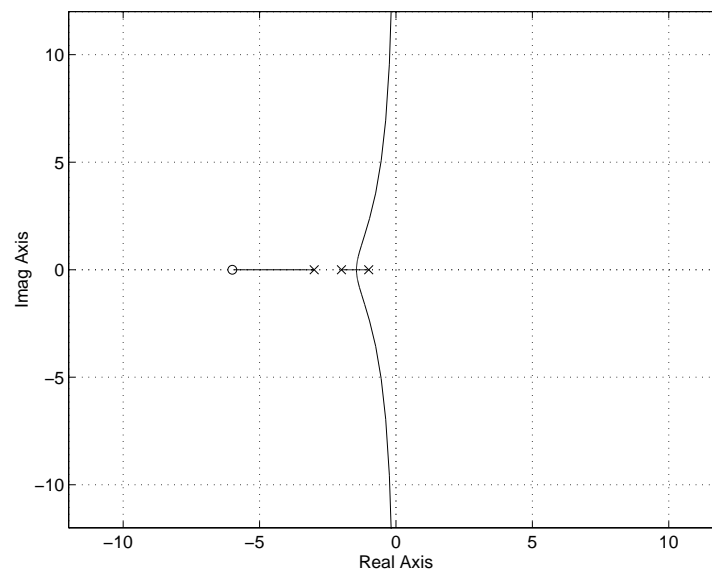


Figure 9.20: Root locus of the original system

### 9.4.5 Compensator Design with Phase-Lag Controller

Compensator design using phase-lag controllers is based on the compensator's attenuation at high frequencies, which causes a shift of the gain crossover frequency to the lower frequency region where the phase margin is high. The phase-lag compensator can be designed by the following algorithm.

**Algorithm 9.2:**

1. Determine the value of the Bode gain  $K_B$  that satisfies the steady state error requirement.
2. Find on the phase Bode plot the frequency which has the phase margin equal to the desired phase margin increased by  $5^\circ$  to  $10^\circ$ . This frequency represents the new gain crossover frequency,  $\omega_{cgnew}$ .
3. Read the required attenuation at the new gain crossover frequency, i.e.  $|\Delta G(j\omega_{cgnew})|_{dB}$ , and find the parameter  $a$  from

$$-20 \log \left( \frac{p_1}{z_1} \right) = -20 \log (a) = |\Delta G(j\omega_{cgnew})|_{dB}$$

which implies

$$a = 10^{-\frac{1}{20}|\Delta G(j\omega_{cgnew})|_{dB}} = \frac{1}{|\Delta G(j\omega_{cgnew})|}$$

Note that

$$|\Delta G(j\omega_{cgnew})| = \frac{|K||j\omega_{cgnew} + z_1||j\omega_{cgnew} + z_2|\cdots}{|j\omega_{cgnew} + p_1||j\omega_{cgnew} + p_2|\cdots}$$

4. Place the controller zero one decade to the left of the new gain crossover frequency, that is

$$z_c = \frac{\omega_{cgnew}}{10}$$

Find the pole location from  $p_c = az_c = a\omega_{cgnew}/10$ . The required compensator has the form

$$G_c(s) = \frac{as + p_c}{s + p_c}$$

5. Redraw the Bode diagram of the given system with the controller and check the values for the gain and phase margins. If they are satisfactory, the controller design is done, otherwise repeat steps 1–5.

**Example 9.5:** Consider a control system represented by

$$G(s) = \frac{K}{s(s+2)(s+30)}$$

Design a phase-lag compensator such that the following specifications are met:  $e_{ssramp} \leq 0.05$ ,  $Pm \geq 45^\circ$ . The minimum value for the static gain that produces the required steady state error is equal to  $K = 1200$ . The original system with this static gain has phase and gain margins given by  $Pm = 6.6449^\circ$ ,  $Gm = 4.0824$  dB and crossover frequencies of  $\omega_{cg} = 6.1031$  rad/s,  $\omega_{cp} = 7.746$  rad/s.

The new gain crossover frequency can be estimated as  $\omega_{cgnew} = 1.4$  rad/s since for that frequency the phase margin of the original system is approximately  $50^\circ$ . At  $\omega_{cgnew} = 1.4$  rad/s the required gain attenuation is obtained by MATLAB as

```

wcgnew=1.4;
d1=1200;
g1=abs(j*1.4);
g2=abs(j*1.4+2);
g3=abs(j*1.4+30);
dG=d1/(g1*g2*g3);

```

which produces  $|\Delta G(j1.4)| = 11.6906$  and  $a = 1/|\Delta G(j1.4)| = 0.0855$ . The compensator's pole and zero are obtained as  $-z_c = -\omega_{cgnew}/10 = -0.14$  and  $-p_c = -a\omega_{cgnew}/10 = -0.0120$  (see step 4 of Algorithm 9.2). The transfer function of the phase-lag compensator is

$$G_c(s) = \frac{0.0855s + 0.0120}{s + 0.0120}$$

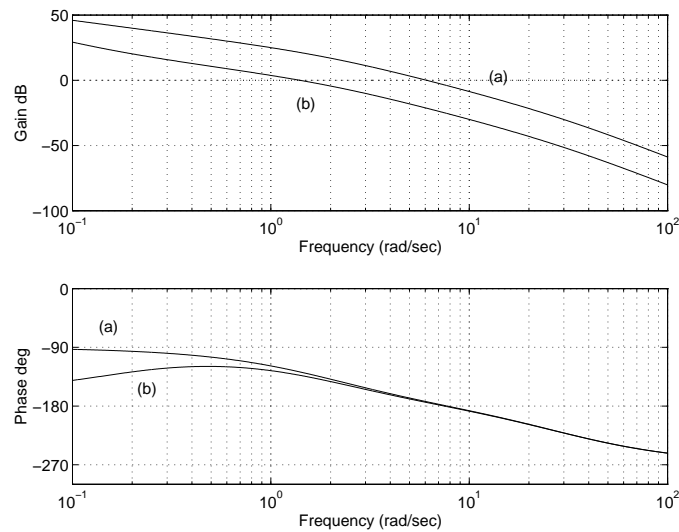


Figure 9.21: Bode diagrams for the original system (a) and compensated system (b) of Example 9.5

The new phase and gain margins and the actual crossover frequencies are  $Pmc = 47.03^\circ$ ,  $Gmc = 24.82\text{ dB}$ ,  $\omega_{cgnew} = 1.405\text{ rad/s}$ ,  $\omega_{cpnew} = 7.477\text{ rad/s}$  and so the design requirements are satisfied. The step responses of the original and compensated systems are presented in Figure 9.22.

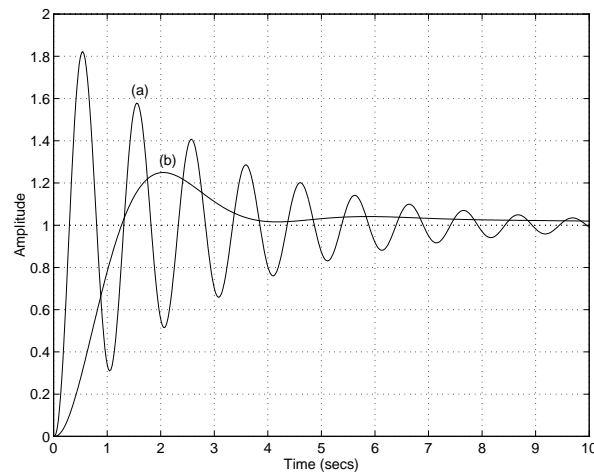


Figure 9.22: Step responses for the original system (a) and compensated system (b) of Example 9.5

It can be seen from this figure that the overshoot is reduced from roughly 0.83 to 0.3. In addition, it can be observed that the settling time is also reduced. Note that the phase-lag controller reduces the system bandwidth ( $\omega_{cgnew} < \omega_{cg}$ ) so that the rise time of the compensated system is increased.

◇



### 9.4.6 Compensator Design with Phase-Lag-Lead Controller

Compensator design using a phase-lag-lead controller can be performed according to the algorithm given below, in which we first form a phase-lead compensator and then a phase-lag compensator. Finally, we connect them together in series.

**Algorithm 9.3:**

1. Set a value for the static gain  $K_B$  such that the steady state error requirement is satisfied.
2. Draw Bode diagrams with  $K_B$  obtained in step 1 and find the corresponding phase and gain margins.
3. Find the difference between the actual and desired phase margins,  $\Delta\phi = Pmd - Pm$ , and take  $\phi_{max}$  to be a little bit greater than  $\Delta\phi$ . Calculate the parameter  $a_2$  of a phase-lead controller by using formula (9.36), that is

$$a_2 = \frac{1 + \sin \phi_{max}}{1 - \sin \phi_{max}}$$

4. Locate the new gain crossover frequency at the point where

$$20 \log |G(j\omega_{cgnew})| = -10 \log a_2 \quad (9.37)$$

5. Compute the values for the phase-lead compensator's pole and zero from

$$p_{c2} = \omega_{cgnew} \sqrt{a_2}, \quad z_{c2} = p_{c2} / a_2 \quad (9.38)$$

6. Select the phase-lag compensator's zero and pole according to

$$z_{c1} = 0.1 z_{c2}, \quad p_{c1} = z_{c1} / a_2 \quad (9.39)$$

7. Form the transfer function of the phase-lag-lead compensator as

$$G_c(s) = G_{lag}(s) \times G_{lead}(s) = \frac{s + z_{c1}}{s + p_{c1}} \times \frac{s + z_{c2}}{s + p_{c2}}$$

8. Plot Bode diagrams of the compensated system and check whether the design specifications are met. If not, repeat some of the steps of the proposed algorithm—in most cases go back to steps 3 or 4.

The phase-lead part of this compensator helps to increase the phase margin (increases the damping ratio, which reduces the maximum percent overshoot and settling time) and broaden the system's bandwidth (reduces the rise time). The phase-lag part, on the other hand, helps to improve the steady state errors.

**Example 9.6:** Consider a control system that has the open-loop transfer function

$$G(s) = \frac{K(s + 10)}{(s^2 + 2s + 2)(s + 20)}$$

For this system we design a phase-lag-lead controller by following Algorithm 9.3 such that the compensated system has a steady state error of less than 4% and a phase margin greater than  $50^\circ$ . In the first step, we choose a value for the static gain  $K$  that produces the desired steady state error. It is easy to check that  $K = 100 \Rightarrow e_{ss} = 3.85\%$ , and therefore in the following we stick with this value for the static gain. Bode diagrams of the original system with  $K = 100$  are presented in Figure 9.23.

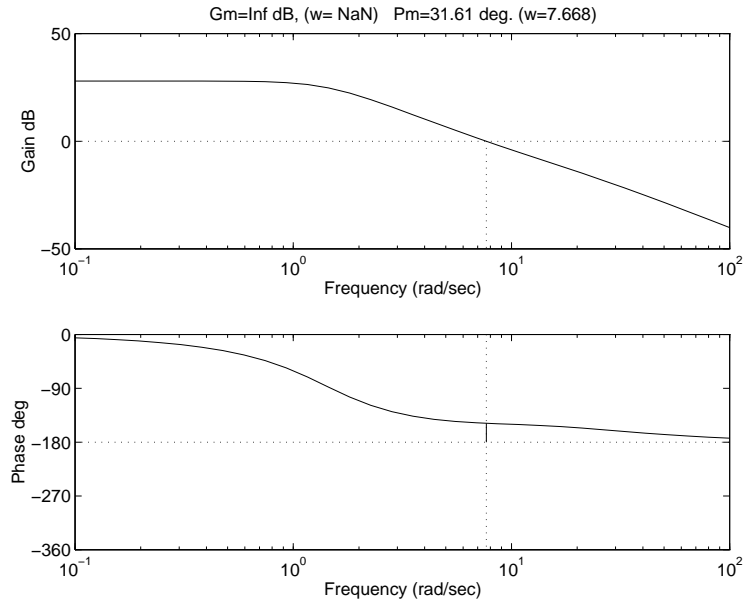


Figure 9.23: Bode diagrams of the original system

It can be seen from these diagrams—and with help of MATLAB determined accurately—that the phase and gain margins and the corresponding crossover frequencies are given by  $Pm = 31.61^\circ$ ,  $Gm = \infty$ , and  $\omega_{cg} = 7.668 \text{ rad/s}$ ,  $\omega_{cp} = \infty$ . According to step 3 of Algorithm 9.3, a controller has to introduce a phase lead of  $18.39^\circ$ . We take  $\phi_{max} = 25^\circ$  and find the required parameter  $a_2 = 2.4639$ . Taking  $\omega_{cgnew} = 20 \text{ rad/s}$  in step 4 and completing the design steps 5–8 we find that  $Pm = 39.94^\circ$ , which is not satisfactory. We go back to step 3 and take  $\phi_{max} = 30^\circ = 0.5236 \text{ rad}$ , which implies  $a_2 = 3$ .

Step 4 of Algorithm 9.3 can be executed efficiently by MATLAB by performing the following search. Since  $-10 \log 3 = -10.9861 \text{ dB}$  we search the magnitude diagram for the frequency where the attenuation is approximately equal to  $-11 \text{ dB}$ . We start search at  $\omega = 20 \text{ rad/s}$  since at that point, according to Figure 9.23, the

attenuation is obviously smaller than  $-11$  dB. MATLAB program used

```
w=20;
while 20*log10(100*abs(j*w+10)/
abs(((j*w)^2+2*j*w+2)*(j*w+20)))<-11;
w=w-1;
end
```

This program produces  $\omega_{cnew} = 10$  rad/s. In steps 5 and 6 the phase-lag-lead controller zeros and poles are obtained as  $-p_{c2} = -17.3205$ ,  $-z_{c2} = -5.7735$  for the phase-lead part and  $-p_{c1} = -0.1925$ ,  $-z_{c1} = -0.5774$  for the phase-lag part; hence the phase-lag-lead controller has the form

$$G_c(s) = \frac{s + 0.5774}{s + 0.1925} \times \frac{s + 17.3205}{s + 5.7735}$$

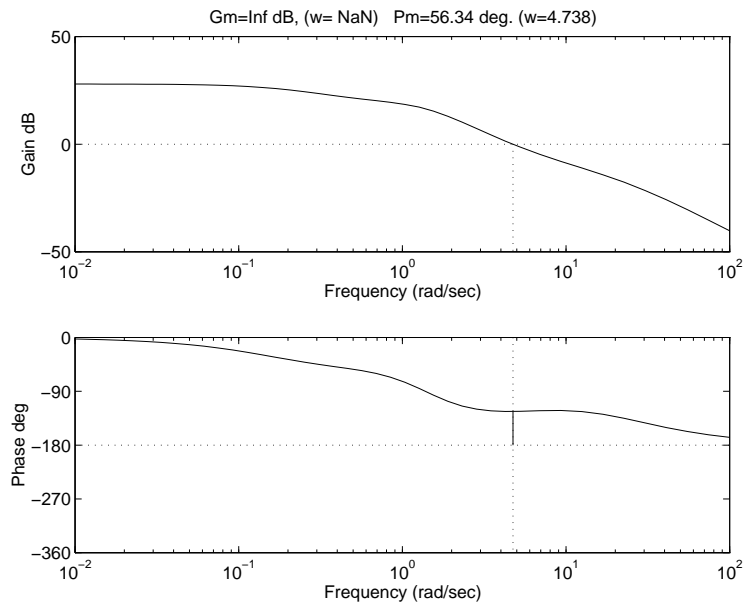


Figure 9.24 Bode diagrams of the compensated system

It can be seen that the phase margin obtained of  $56.34^\circ$  meets the design requirement and that the actual gain crossover frequency,  $4.738 \text{ rad/s}$ , is considerably smaller than the one predicted. This contributes to the generally accepted inaccuracy of frequency methods for controller design based on Bode diagrams.

The step responses of the original and compensated systems are compared in Figure 9.25. The transient response of the compensated system is improved since the maximum percent overshoot is considerably reduced. However, the system rise time is increased due to the fact that the system bandwidth is shortened ( $\omega_{cgnew} = 4.738 \text{ rad/s} < \omega_{cg} = 7.668 \text{ rad/s}$ ).

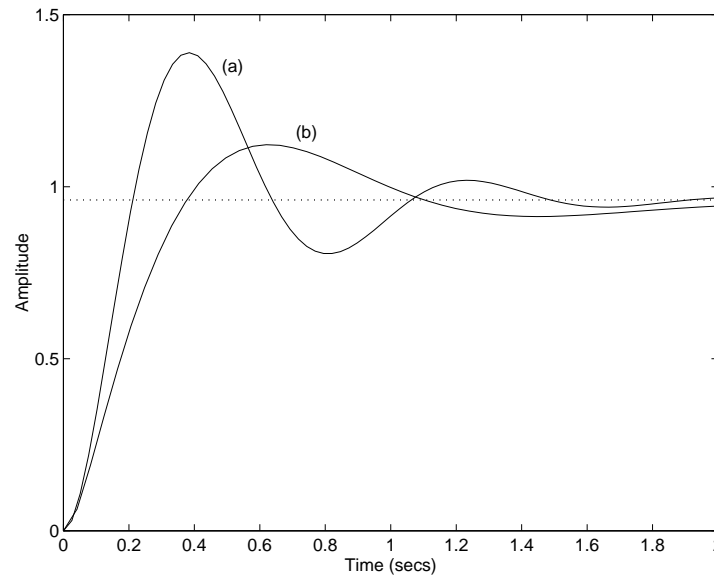


Figure 9.25 Step responses of the original (a) and compensated (b) systems

## 9.5 MATLAB Case Study

Consider the problem of finding a controller for the ship positioning control system given in Problem 7.5. The goal is to increase stability phase margin above  $30^\circ$ . The problem matrices are given by

$$\mathbf{A} = \begin{bmatrix} -0.0546 & 0 & 0.5435 \\ 1 & 0 & 0 \\ 0 & 0 & -1.55 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1.55 \end{bmatrix}, \quad \mathbf{C} = [0 \quad 1 \quad 0], \quad \mathbf{D} = 0$$

The transfer function of the ship positioning system is obtained by the MATLAB instruction `[num,den]=ss2tf(A,B,C,D)` and is given by

$$G(s) = \frac{0.8424}{s(s + 1.55)(s + 0.0546)}$$

The phase and gain stability margins of this system are  $Pm = -19.94^\circ$  and  $Gm = -15.86$  dB, with the crossover frequencies  $\omega_{cp} = 0.2909$  rad/s and  $\omega_{cg} = 0.7025$  rad/s (see the Bode diagrams in Figure 9.26). From known values for the phase and gain margins, we can conclude that this system has very poor stability properties.

Since the phase margin is well below the desired one, we need a controller which will make up for almost a  $50^\circ$  increase in phase. In general, it is hard to stabilize systems that have large negative phase and gain stability margins. In the following we will design phase-lead, phase-lag, and phase-lag-lead controllers to solve this problem and compare the results obtained.

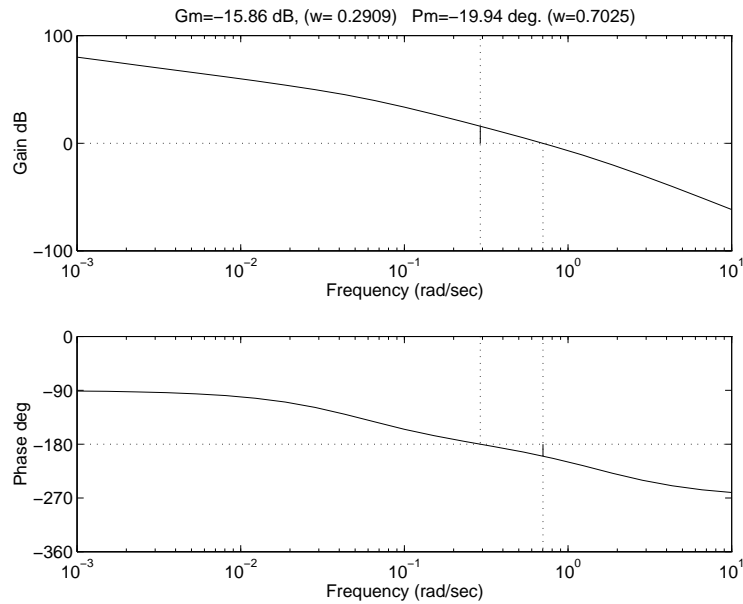


Figure 9.26: Bode diagrams of a ship positioning control system

*Phase-Lead Controller:* By using Algorithm 9.1 with  $\phi_{max} = 50^\circ + 10^\circ = 60^\circ$  we get a phase margin of only  $23.536^\circ$ , which is not satisfactory. It is necessary to make up for  $\phi_{max} = 50^\circ + 27^\circ = 87^\circ$ . In the latter case the compensator has the transfer function

$$G_c(s) = 76.31 \frac{s + 0.2038}{s + 15.55}$$

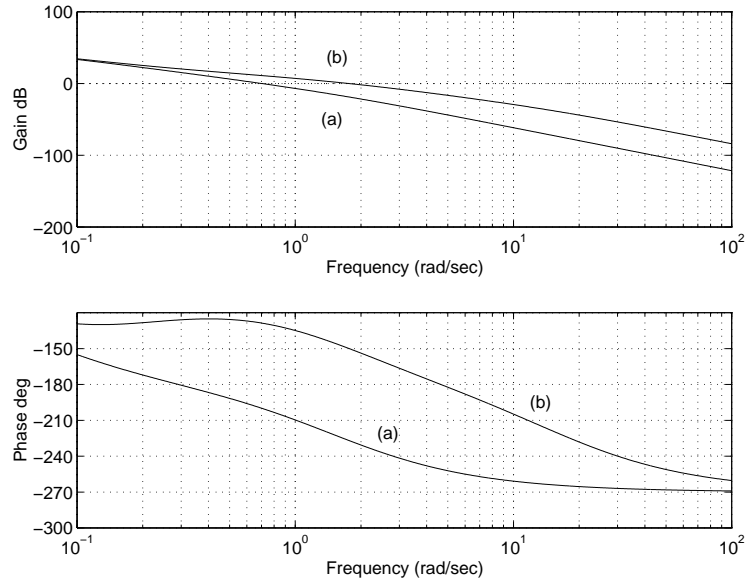


Figure 9.27: Bode diagrams for a ship positioning system:  
(a) original system, (b) phase-lead compensated system

The gain and phase stability margins of the compensated system are found from the above Bode diagrams as  $Gmc = 15.1603$  dB,  $Pmc = 30.0538^\circ$ , and the crossover frequencies are  $\omega_{cpc} = 1.7618$  rad/s,  $\omega_{cgc} = 4.6419$  rad/s. The step response of the compensated system exhibits an overshoot of 45.47% (see Figure 9.28).

*Phase-Lag-Lead Controller:* By using Algorithm 9.3 we find the compensator transfer function as

$$G_c(s) = 7.524 \frac{s + 0.1599}{s + 1.203} \times 0.1329 \frac{s + 0.016}{s + 0.002125}$$



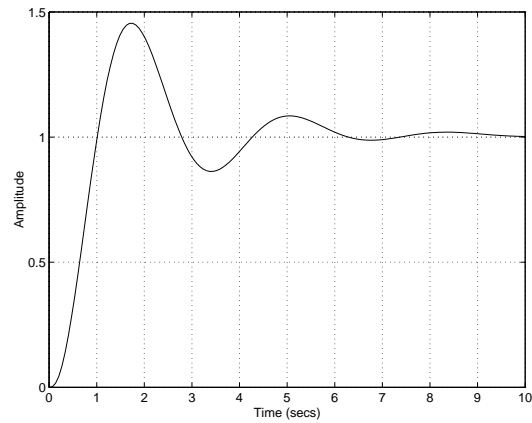


Figure 9.28: Step response of the compensated system with a phase-lead controller

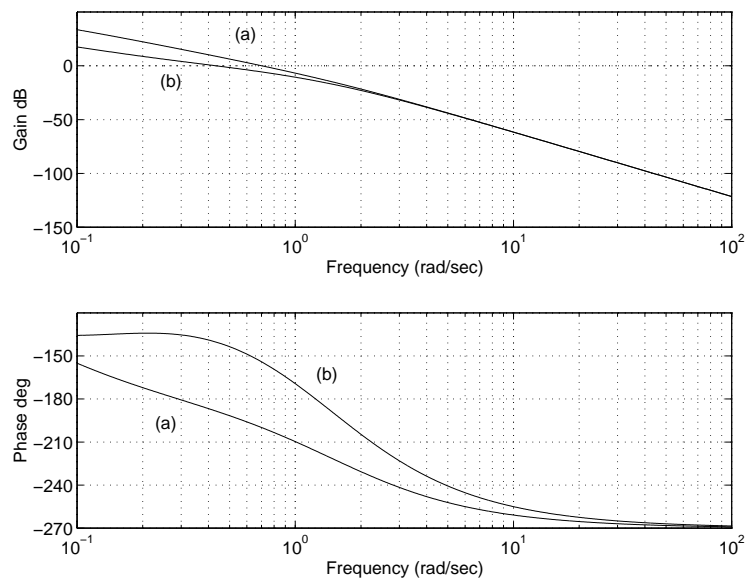


Figure 9.29: Bode diagrams for a ship positioning control system:  
(a) original system, (b) phase-lag-lead compensated system

The phase and gain margins of the compensated system are given by  $Pmc = 39.6694^\circ$ ,  $Gmc = 14$  dB and the crossover frequencies are  $\omega_{cgc} = 0.4332$  rad/s,  $\omega_{cpc} = 1.2401$  rad/s.

From the step response of the compensated system (see Figure 9.30), we can observe that this compensated system has a smaller overshoot and a larger rise time than the system compensated only by the phase-lead controller.

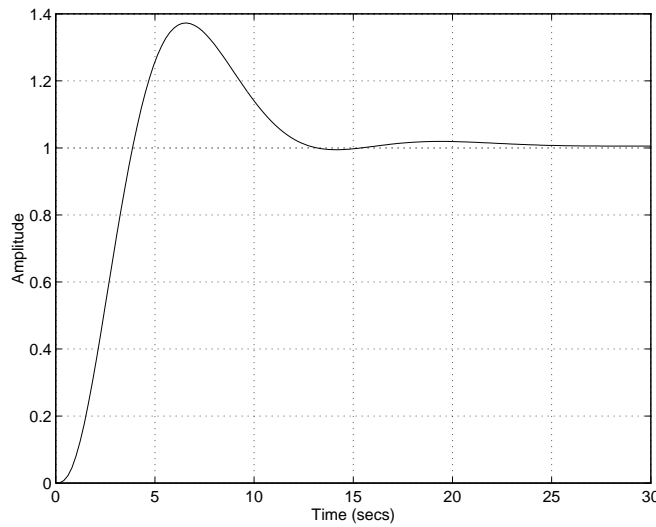


Figure 9.30: Step response of the compensated system with a phase-lag-lead controller

*Phase-Lag Controller:* If we choose a new gain crossover frequency at  $\omega_{cgnew} = 0.03$  rad/s, the phase margin at that point will clearly be above  $50^\circ$ . Proceeding with a phase-lag compensator design, according to Algorithm 9.2, we get  $|\Delta G(j0.03)| = 290.7390$  and  $a = 0.034$ , which implies  $z_c = 0.003$  and  $p_c = 1.0319 \times 10^{-5}$ . Using the corresponding phase-lag compensator produces very good stability margins for the compensated system, i.e.  $Gm = 32.91$  dB and  $Pm = 54.33^\circ$ . The maximum percent overshoot obtained is much better than with the previously used compensators and is equal to  $MPOS = 18\%$ . However, the closed-loop step

response reveals that the obtained system is too sluggish since the response peak time is  $t_p = 95.2381$  s (note that in the previous two cases the peak time is only a few seconds).

One may try to get better agreement by designing a phase-lag compensator, which will reduce the phase margin of the compensated system to just above  $30^\circ$ . In order to do this we write a MATLAB program, which searches the phase Bode diagram and finds the frequency corresponding to the prespecified value of the phase. That frequency is used as a new gain crossover frequency. Let  $Pm = 35^\circ = 0.6109$  rad. The MATLAB program is

```
w=0.1;
while pi+
angle(1/((j*w)*(j*w+1.55)*(j*w+0.0546)))<0.6109;
w=w-0.01;
end
dG=0.8424*abs(1/((j*w)*(j*w+1.55)*(j*w+0.0546)));
```

This program produces  $\omega_{cgnew} = 0.07$  rad/s and  $|\Delta G(j0.07)| = 87.3677$ . From step 4 of Algorithm 9.2 we obtain the phase-lag controller of the form

$$G_c(s) = \frac{s + 0.009}{s + 0.000081} \times 0.0114$$

The Bode diagrams of the compensated system are given in Figure 9.31.

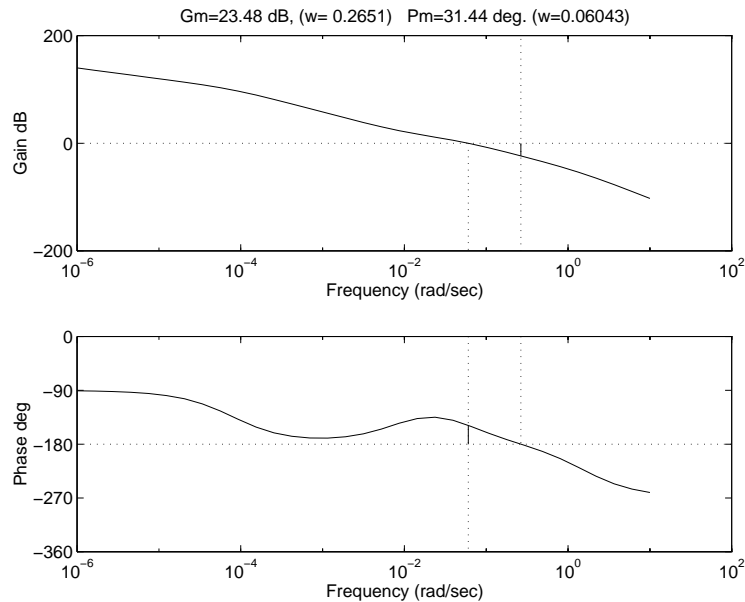


Figure 9.31: Bode diagram of the phase-lag compensated system

It can be seen that the phase and gain margins are satisfactory and given by  $Pm = 31.44^\circ$  and  $Gm = 23.48$  dB. The actual gain crossover frequencies are  $\omega_{cgnew} = 0.06043$  rad/s and  $\omega_{cpnew} = 0.2651$  rad/s.

The closed-loop step response of the phase-lag compensated system, given in Figure 9.32, shows that the peak time is reduced to  $t_p = 50.15$  s—which is still fairly big—and that the maximum percent overshoot is increased to  $MPOS = 45.82\%$ , which is comparable to the phase-lead and phase-lag-lead compensation.

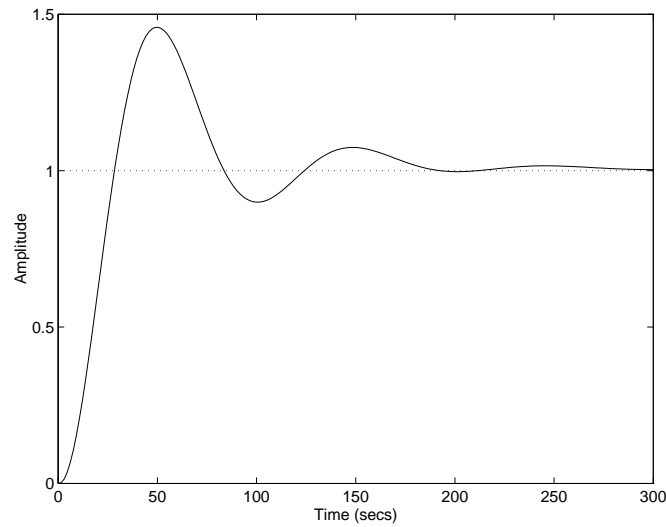


Figure 9.32: Step response of the phase-lag compensated system

Comparing all three controllers and their performances, we can conclude that, for this particular problem, the phase-lag compensation produces the worst result, and therefore either the phase-lead or phase-lag-lead controller should be used.

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